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REPEATED CASCADE THEORY OF GRAVITY TURBULENCE WITH AN INTERFACE

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GRAVITY TURBULENCE CONNECTED WITH INTERFACES

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ABSTRACT

The spectral distributions of turbulence, as generated by the gravity waves on the interface between two fluids, are investigated. Both stable and unstable surfaces are considered. An unstable surface refers to the early development of turbulence from the Taylor instability. A stable surface may refer to the sea surface. A repeated cascade method is used to close the hierarchy of correlations at their fourth order, and to describe the eddy transport property as a memory chain of eddy relaxations. The production, inertia and dissipation subranges of spectral distributions on an unstable surface with friction are found to follow the laws k^{-2}, k^{-3}, and k^{-3} for the kinetic energy, and $k^{-3.5}$, k^{-1}, and k^{-5} for the surface elevation. The inertia, eddy dissipation by gravity, and molecular dissipation subranges on a stable surface with friction are found to follow the law k^{-3} for the kinetic energy, and the laws k^{-1}, k^{-5} and k^{-5} for the surface elevation, respectively. The spectra with surface tension are also investigated. The physical parameters and the numerical coefficients are determined analytically.

I. INTRODUCTION

For neutral fluids, we can distinguish hydrodynamic turbulence and gravity turbulence. The latter may occur in a free medium, e.g. in atmospheres and oceans with a mean gradient in density or temperature. It can also appear on a stable, or an unstable, surface, of a heavy liquid below, or above, a light one. For example, the sea surface is such a stable surface. The turbulent motions in a stable background may require an external source of energy for their maintenance. On the other hand, an unstable surface may eventually develop into singular fingers, broken boundaries and droplets which cease to constitute a continuous surface. Therefore, the gravity turbulence can only refer to the early stage of the development of a continuous turbulent surface under an unstable condition.

Mathematically speaking, when the pressure is eliminated between the Navier-Stokes equation and the equation of continuity, the hydrodynamic turbulence is described by one single equation of motion, and the gravity turbulence is described by a system of two equations, governing the velocity and temperature, or density, in a free medium, or the velocity and the elevation on a moving surface.

In view of the complexity of the dynamical equations, the dimensional method had been relied upon for solutions. It enabled Kolmogoroff[1] and Heisenberg[2] to derive the spectral laws for the velocity fluctuations,

$$F(k) = \text{const}\ \varepsilon^{2/3} k^{-5/3}, \qquad F(k) = \text{const}\,(\varepsilon/\nu^2)\, k^{-7}. \tag{1}$$

in the inertia and dissipation subranges of a hydrodynamic turbulence. On the same basis, Shur[3] proposed the spectrum

$$F = \text{const}\ N^2 k^{-3} \tag{2}$$

for the gravity turbulence in a free medium, and Phillips[4] obtained an energy spectrum

$$F = \text{const}\ g\, k^{-2} \tag{3}$$

and the elevation spectra

$$H(k) = \text{const}\ k^{-3}, \qquad H(\omega) = \text{const}\ g^2 \omega^{-5} \tag{4}$$

in gravity turbulence with surfaces. The spectra may be determined in the space of wave number k, or of frequency ω. The parameters in the above formulas are: the rate of energy dissipation ε, the kinematic viscosity ν, the acceleration of gravity g, and the Brunt-Väisälä frequency N representative of a mean gradient in temperature or density.

It is obvious that the spectra should not be identical for both stable and unstable surfaces. Since the dimensional analysis only recognizes the parameters, without going into dynamical mechanisms, it is not able to determine the conditions, stable or unstable, under which the dimensional laws (2), (3) and (4) should apply, although the authors intended for their use under a stable condition.

In view of the above difficulties, it is necessary to consider an analytic treatment. Since most analytic theories in hydrodynamic turbulence cannot even predict satisfactorily a

Kolmogoroff law (1) which requires only one parameter, we shall resort to the method of repeated cascade, proposed by Tchen[5]. It enables closing the hierarchy of correlations to their fourth order, and characterizing a turbulent transport property by means of a memory chain of eddy relaxations. For a hydrodynamic turbulence, it derived the Kolmogoroff law of turbulence in the inertia subrange, and a k^{-1} law in a gradient flow. In the present paper, we shall extend the method to the gravity turbulence on stable and unstable surfaces. The gravity turbulence in a free medium will not be treated here.

II. DERIVATION OF THE EQUATIONS FOR THE MOTION OF THE SURFACE

The equations of Navier-Stokes and of continuity for the motion of a liquid are

$$\frac{\partial v_i}{\partial t} + v_j \frac{\partial v_i}{\partial x_j} = -\frac{1}{\rho}\frac{\partial p}{\partial x_i} - g\,\delta_{i3} - \gamma v_i, \tag{5}$$

$$\frac{\partial v_i}{\partial x_i} = 0, \qquad i, j = 1, 2, 3. \tag{6}$$

where $v_i(t, x_1, x_2, x_3)$ is a velocity vector with components in three directions x_1, x_2, x_3, p is the pressure, ρ is the density, γ is a frictional coefficient, and g is the acceleration of gravity.

We introduce a velocity potential $\phi(t, x_1, x_2, x_3)$ such that

$$v_i = \frac{\partial \phi}{\partial x_i}, \qquad i = 1, 2, 3, \tag{7}$$

and transform Eqs. (5) and (6) into a Bernoulli equation and a Laplace equation,

$$\frac{\partial \phi}{\partial t} + \frac{1}{2} v_i^2 + \frac{p}{\rho} + g x_3 + \gamma \phi = f(t), \tag{8}$$

$$\frac{\partial^2 \phi}{\partial x_i^2} = 0, \qquad i = 1, 2, 3, \tag{9}$$

respectively, where $f(t)$ is a function of time, which may be determined by a condition at the surface, if necessary.

In order to apply Eqs. (8) and (9) to the surface, we put $x_3 = \zeta(t, x_1, x_2)$ in (8), and write the velocity, potential and pressure on the surface, as follows:

$$u_i(t, x_1, x_2) = v_i\left[t, x_1, x_2, x_3\ \zeta(t, x_1, x_2)\right], \quad i = 1, 2, \tag{10}$$

$$\varphi(t, x_1, x_2) = \phi(t, x_1, x_2, x_3 = \zeta), \tag{11}$$

$$p_s(t, x_1, x_2) = p(t, x_1, x_2, x_3 = \zeta) = constant, \tag{12}$$

transforming (8) into

$$\left(\frac{\partial \phi}{\partial t}\right)_{x_3=\zeta} + \frac{1}{2}(u_1^2 + u_2^2) + \frac{1}{2}\left[v_3(t, x_1, x_2, x_3 = \zeta)\right]$$

$$+ \frac{p_s}{\rho} - \frac{T}{\rho}\nabla^2\zeta + g\zeta + \gamma\varphi = f(t), \tag{13}$$

where T is a surface tension.

It can be noted that p_s and f, which appear in (13), can be eliminated by taking a gradient, giving

$$\frac{\partial}{\partial x_i}\left(\frac{\partial \phi}{\partial t}\right)_{x_3=\zeta} + \frac{1}{2}\frac{\partial u_j^2}{\partial x_i} + \frac{1}{2}\frac{\partial}{\partial x_i}\left[v_3(t, x_1, x_2, x_3 = \zeta)\right]^2$$

$$= \frac{T}{\rho}\frac{\partial^3 \zeta}{\partial x_i \partial x_j^2} - g\frac{\partial \zeta}{\partial x_i} - \gamma\frac{\partial \varphi}{\partial x_i}, \quad i\ 1, 2. \tag{14}$$

Equation (14), combined with the following kinematic equation for the surface elevation ζ,

$$\frac{d\zeta}{dt} = \left(\frac{\partial}{\partial t} + u\nabla\right)\zeta$$

$$= v_3(t, x_1, x_2, x_3 = \zeta), \tag{15}$$

forms two fundamental equations determining the motions of the surface. Here

$$v_3(t, x_1, x_2, \zeta) = -\xi \int_{\zeta, -\infty}^{\infty, \zeta} dx_3 \left(\frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2} \right) . \tag{16}$$

The limits of integration (ζ,∞) refer to an unstable surface, i.e. with $\xi = +1$, and the limits $(-\infty,\zeta)$ refer to a stable surface, i.e. with $\xi = -1$. Equation (15) may be regarded as originated from the integration of an equation of mass conservation.

Since certain transformations are necessary to simplify Eqs. (14) and (15), we write the potential in Fourier transform

$$\phi(t, x_1, x_2, x_3) = \iiint_{-\infty}^{\infty} d\kappa_1\, d\kappa_2\, d\kappa_3\; \phi(t, \kappa_1, \kappa_2, \kappa_3) \exp[i(\kappa_1 x_1 + \kappa_2 x_2 + \kappa_3 x_3)] \tag{17}$$

where the wave number κ satisfies the relation

$$\kappa_1^2 + \kappa_2^2 + \kappa_3^2 = 0 , \tag{18}$$

as a consequence of the equation of continuity (9).

As a surface wave decreases its amplitude vertically like

$$\exp(-\xi k x_3) ,$$

we find

$$\kappa_3 = i \xi k \tag{19}$$

with a real k as a consequence of the incompressibility of fluid, as expressed by the relation (18). This permits rewriting (17) in the form

$$\phi(t,\kappa_1,\kappa_2,\kappa_3) = \phi(t,\kappa_1,\kappa_2)\,\delta(\kappa_3 - i\xi k) \tag{20}$$

and reducing formally (17) to

$$\varphi(t,x_1,x_2) = \iint_{-\infty}^{\infty} d\kappa_1\, d\kappa_2\; \phi(t,\kappa_1,\kappa_2) \exp[i(\kappa_1 x_1 + \kappa_2 x_2)] \exp(-\xi k \zeta) \tag{21}$$

$$u_i(t,x_1,x_2) = \iint_{-\infty}^{\infty} d\kappa_1\, d\kappa_2\; \phi(t,\kappa_1,\kappa_2)\, i\kappa_i \exp[i(\kappa_1 x_1 + \kappa_2 x_2)] \exp(-\xi k \zeta),\; i = 1,2 \tag{22}$$

$$v_3(t,x_1,x_2,x_3) = -\xi \iint_{-\infty}^{\infty} d\kappa_1\, d\kappa_2\; \phi(t,\kappa_1,\kappa_2)\, k \exp[i(\kappa_1 x_1 + \kappa_2 x_2)] \exp(-\xi k \zeta) \tag{23}$$

$$v_3(t,x_1,x_2,x_3=\zeta) = -\xi \iint_{-\infty}^{\infty} d\kappa_1\, d\kappa_2\; \phi(t,\kappa_1,\kappa_2)\, k \exp[i(\kappa_1 x_1 + \kappa_2 x_2)] \exp(-\xi k \zeta) \tag{24}$$

Those formal relations provides the possibility of transforming all quantities related to velocity and potential in (14) and (15) into $\underset{\sim}{u}$, as follows:

$$\frac{\partial}{\partial x_i}\left(\frac{\partial \phi}{\partial t}\right)_{x_3=\zeta} = \frac{\partial u_i}{\partial t} + R_1 \tag{25}$$

$$\frac{1}{2}\frac{\partial u_j^2}{\partial x_i} = u_j \frac{\partial u_i}{\partial x_j} + R_2 \tag{26}$$

$$\gamma \frac{\partial \varphi}{\partial x_i} = \gamma u_i + R_3 \tag{27}$$

$$v_3(t,x_1,x_2,x_3=\zeta) = i\xi \iint_{-\infty}^{\infty} dk_1\, dk_2\;\; k^{-1} k_j\, u_j(t,k) \exp\left[i(k_1 x_1 + k_2 x_2)\right] + R_4 \tag{28}$$

Here and in the following, the indices refer to i, j = 1, 2. Furthermore, the functions R_1, R_2, R_3 and R_4 are corrections proportional to ζ and its derivatives, which in turn are proportional to $\underset{\sim}{k}\cdot\underset{\sim}{u}(\underset{\sim}{k})$.

We shall now resort to the Boussinesq approximation, well known in problems of gravity waves, which neglects such terms as proportional to ζ or to $\underset{\sim}{k}\cdot\underset{\sim}{u}$, except the term $g\nabla\zeta$ associated with the gravity g as a driving force in the momentum equation. That approximation amounts to neglecting all the above mentioned correction functions, and thereby reducing (14) to the following simpler form in $\underset{\sim}{x}$-space,

$$\left(\frac{\partial}{\partial t} + \underset{\sim}{u}\cdot\nabla\right)\underset{\sim}{u} + \gamma\underset{\sim}{u} = -\nabla\left(g^{*}\zeta\right) = \underset{\sim}{E}(\underset{\sim}{x}) , \tag{29}$$

or, in Fourier space,

$$\frac{\partial u_i(k)}{\partial t} + \int_{-\infty}^{\infty} d\underset{\sim}{k}'\, i k_j'\, u_j(\underset{\sim}{k}-\underset{\sim}{k}')\, u_i(\underset{\sim}{k}') + \gamma u_i(\underset{\sim}{k})$$

$$= -ig^{*}(k)\, k_i\, \zeta(\underset{\sim}{k}) = E_i(\underset{\sim}{k}) , \tag{30}$$

where

$$g^{*}(x) = g\left(1 - \frac{T}{\rho g}\nabla^2\right) \tag{31}$$

$$g^{*}(k) = g\left(1 + \frac{T}{\rho g}k^2\right) \tag{32}$$

and $\underset{\sim}{E}$ is a driving force due to the gravitation pull of the surface. The same approximation reduces (15) to the following form in Fourier space:

$$\frac{\partial \zeta(\underset{\sim}{k})}{\partial t} + \int_{-\infty}^{\infty} d\underset{\sim}{k}'\, i k_j'\, u_j(\underset{\sim}{k}-\underset{\sim}{k}')\, \zeta(\underset{\sim}{k}')$$

$$= \xi i k^{-1} k_j u_j(\underset{\sim}{k}) \quad (33)$$

Here $\underset{\sim}{k} = (k_1, k_2)$ is a wave number vector in two dimensions; the dependence of $\underset{\sim}{u}$ and ζ with time is understood. In the following we shall keep the dynamical equations for the surface in their Fourier form, because the right hand side of (33) does not possess a simple form of inversion.

Here and in the following we omit the writing of the variable t in the argument of all time dependent functions, except when a need for specific distinction arises.

For convenience, we define a speed of propagation

$$c(k) = (g^*/k)^{\frac{1}{2}} , \quad (34)$$

and a drift velocity of the surface, called "potential drift," in k-space

$$w(\underset{\sim}{k}) = c k \zeta(\underset{\sim}{k}) \quad (35)$$

With those notations (34) and (35), we transform (30) and (33) to a more symmetry form:

$$\frac{\partial u_i(\underset{\sim}{k})}{\partial t} + \gamma u_i(\underset{\sim}{k}) + \int_{-\infty}^{\infty} d\underset{\sim}{k}'\, i k_j'\, u_j(\underset{\sim}{k}-\underset{\sim}{k}')\, u_i(\underset{\sim}{k}')$$

$$= -ic k_i w(\underset{\sim}{k}) = E_i(\underset{\sim}{k}) \quad (36)$$

$$\frac{\partial w(\mathbf{k})}{\partial t} + \int_{-\infty}^{\infty} d\mathbf{k}' \; i k_j \, (k/k') \; u_j(\mathbf{k}-\mathbf{k}') \, w(\mathbf{k}')$$

$$= i\xi \, i k_j \, u_j(\mathbf{k}) . \qquad (37)$$

The form represented by (36) and (37) has the advantage that the gravitational coupling

$$\Gamma(k) = \mathbf{E}(\mathbf{k}) \cdot \mathbf{u}(-\mathbf{k})$$

$$= \mathbf{E}(-\mathbf{k}) \cdot \mathbf{u}(\mathbf{k})$$

is of equal magnitude but opposite signs in the equation of evolution of kinetic energy and potential energy

$$\frac{1}{2} \mathbf{u}(\mathbf{k}) \cdot \mathbf{u}(-\mathbf{k}) , \qquad \frac{1}{2} w(\mathbf{k}) \, w(-\mathbf{k}) \qquad (39)$$

for the case of a stable configuration, i.e. $\xi = -1$.

In applications, we shall use both systems of equations, Eqs. (30) and (33), or Eqs. (36) and (37). The letter system will be used, when the mechanism of turbulence is controlled by the gravitational coupling between the kinetic energy and the potential energy, while the former system will be used in other mechanisms where such a coupling does not come into play.

III. OUTLINE OF THE SINGLE AND REPEATED CASCADE METHODS OF TURBULENCE

IIIA. Purposes of Single and Repeated Cascades

In the past, analytical theories of turbulence have been based upon investigating the hierarchy of velocity correlations of various orders, and designing the methods of closure. The degeneration of correlations from high into lower orders can only be performed in an ambiguous way, since all velocities in the high order correlations have to be treated uniformly. It is well known that such a practice may lead to serious difficulties. That those velocities appear uniformly is a certain consequence of the mathematical model adopted. Physically, since a velocity consists of many scales, it is expected that like scales will group together to form correlations of non-uniform pairs playing non-uniform physical roles. In order to allow such a selection of scales in correlations, we prescribe ranks in the velocity variable.

A turbulent motion is a quasi-stationary process, having a continuous spectrum of coupled scales. The large scales form a "macroscopic background," prescribing the background conditions for the motion of small scales. The smaller scales move more "randomly," and, as a result of the statistical effect of their fluctuations, shape up transport properties in the background medium. The above division into macroscopic and random variables are relative to any wave number of the spectrum. Thus we write, for a velocity $\underset{\sim}{u}$ in the physical space or in the wave number space,

$$\underset{\sim}{u} = \underset{\sim}{u}^{\circ} + \underset{\sim}{u}' \tag{40}$$

in a "single cascade," or further decompose

$$\underset{\sim}{u}' = \underset{\sim}{u}^{(1)} + \underset{\sim}{u}^{(2)} + \cdots + \underset{\sim}{u}^{(N)}, \tag{41}$$

so that

$$\underset{\sim}{u} = \underset{\sim}{u}^{\circ} + \cdots + \underset{\sim}{u}^{(N)} \tag{42}$$

becomes a "repeated cascade." More generally, we can decompose $\underset{\sim}{u}$ into

$$\underset{\sim}{u} = \underset{\sim}{V}^{(\alpha)} + \underset{\sim}{v}^{(\alpha)}, \tag{43}$$

instead of (40), with

$$\underset{\sim}{V}^{(\alpha)} = \underset{\sim}{u}^{\circ} + \cdots + \underset{\sim}{u}^{(\alpha-1)} \tag{44}$$

$$\underset{\sim}{v}^{(\alpha)} = \underset{\sim}{u}^{(\alpha)} + \cdots + \underset{\sim}{u}^{(N)} \tag{45}$$

Evidently, $\underset{\sim}{v}^{(1)} = \underset{\sim}{u}'$.

The superscripts denote the ranks, with a higher rank having a higher degree of randomness. Thus the ranks $\underset{\sim}{u}^{\circ}$ and $\underset{\sim}{V}^{(\alpha)}$ will be considered as macroscopic backgrounds, while $\underset{\sim}{u}'$ and $\underset{\sim}{v}^{(\alpha)}$ will be random fluctuations in (40) and (43).

The rank $\underset{\sim}{u}^{\circ}$ is responsible for forming an energy, of scales up to a wave number in the spectrum. The velocity $\underset{\sim}{u}'$ represents all the smaller scales which shape up an eddy viscosity in the background medium. For the purpose of further determining the approach to equilibrium of the transport property, $\underset{\sim}{u}'$ has to be subdivided into higher ranks (41) in a repeated cascade, where $\underset{\sim}{u}^{(1)}$ and $\underset{\sim}{u}^{(2)}$ contribute to an energy and a relaxation frequency required to formulate the eddy viscosity. In continuing the sequence, the higher ranks $u^{(3)}$, $u^{(4)}$ ... relay to high order relaxation frequencies forming a memory chain. The method of closing the hierarchies arising from velocity correlations of various orders, and from the said memory chain has been treated by Tchen in applications to hydrodynamic turbulence[5]. For the similar problem of closing the hierarchies and the memory chain in gravity turbulence, we will need to consider a repeated cascade for the variable $\underset{\sim}{u}$, while a single cascade suffices for

$$\zeta = \zeta^{\circ} + \zeta' \tag{46}$$

and

$$w = w^{\circ} + w' \tag{47}$$

In the above picture of quasi-stationary turbulent process, the large scale motions are considered relatively macroscopic and the smaller scales are more random. Thus, we can associate a high degree of randomness to high wave numbers, by writing

$$u^{(\alpha)}(x) = \int_{k^{(\alpha-1)}}^{k^{(\alpha)}} dk \; u(k) \exp(ik \cdot x) \tag{48}$$

$$\equiv \int_{-\infty}^{\infty} dk \; u^{(\alpha)}(k) \exp(ik \cdot x) \, , \tag{49}$$

$$\begin{aligned} v^{(\alpha)}(x) &= \int_{k^{(\alpha-1)}}^{\infty} dk \; u(k) \exp(ik \cdot x) \\ &\equiv \int_{-\infty}^{\infty} dk \; v^{(\alpha)}(k) \exp(ik \cdot x) \, . \end{aligned} \tag{50}$$

In the notation (49), $u^{(\alpha)}(k)$ is understood to be truncated between the wave number interval $(k^{(\alpha-1)}, k^{(\alpha)})$. The truncation (48) needs not be sharp, and, if necessary, can be regulated by a scaling distribution.

IIIB. Averaging Rules

A "cascade ensemble average," or "rank average," denoted by

$$\langle \cdots \rangle' \tag{51}$$

is expected to separate the two components in (40), by averaging over realizations under identical macroscopic background u°. After such an averaging procedure, the random component u' becomes macroscopically negligible, and the macroscopic component

$\underset{\sim}{u}^{\circ}$ comes out intact. Thus we have

$$\langle \underset{\sim}{u} \rangle' = \underset{\sim}{u}^{\circ}, \qquad \langle \underset{\sim}{u}^{\circ} \rangle' = \underset{\sim}{u}^{\circ}, \qquad \langle \underset{\sim}{u}' \rangle' = 0 \tag{52}$$

Similarly a rank average

$$\langle \cdots \rangle^{\circ} \tag{53}$$

would annul $\underset{\sim}{u}^{\circ}$, i.e.

$$\langle \underset{\sim}{u}^{\circ} \rangle^{\circ} = 0, \tag{54}$$

and could correspond to a spatial average of a length interval X°. It may be noted that X° may tend to infinity in a homogeneous turbulence.

For the use of distinction of a more general rank α, a rank average

$$\langle \cdots \rangle^{(\alpha)} \tag{55}$$

corresponding to a length interval $X^{(\alpha)}$ is introduced. The following averaging rules apply:

$$\langle \underset{\sim}{u}^{(\beta)} \rangle^{(\alpha)} = 0, \qquad \text{if} \quad \beta \geqslant \alpha \tag{56}$$

$$= \underset{\sim}{u}^{(\beta)}, \qquad \text{if} \quad \beta < \alpha \tag{57}$$

as a generalization of (52) and (54).

In Ref. 5 we have introduced a distribution function of many velocities ordered in ranks as random variables, and distribution functions of reduced velocity ranks. They are similar to the distribution functions in the BBGKY hierarchy of kinetic theory of gas. They serve to define the rank average (55). We shall not enter into such a detail here.

IIIC. Ranks and Rank Values of Correlations

In view of the condition of quasi-stationarity of turbulence of rank α, we can write

$$\langle \underset{\sim}{u}^{(\alpha)}(\underset{\sim}{k}') \; \underset{\sim}{u}^{(\alpha)}(\underset{\sim}{k}'') \rangle^{(\alpha)} = \chi^{(\alpha)} \langle \underset{\sim}{u}^{(\alpha)}(\underset{\sim}{k}') \; \underset{\sim}{u}^{(\alpha)}(-\underset{\sim}{k}') \rangle^{(\alpha)} \delta(\underset{\sim}{k}' + \underset{\sim}{k}'') \quad , \tag{58}$$

where

$$\chi^{(\alpha)} = (\pi / X^{(\alpha)})^{-1} \tag{59}$$

is called a "scaling factor," and s = 2 in two dimensions, giving the following relation between the velocity correlations in $\underset{\sim}{k}$-space and $\underset{\sim}{x}$-space:

$$\langle \underset{\sim}{u}^{(\alpha)}(\underset{\sim}{x}) \; \underset{\sim}{u}^{(\alpha)}(\underset{\sim}{x}) \rangle^{(\alpha)} = \int_{-\infty}^{\infty} d\underset{\sim}{k}' \; \chi^{(\alpha)} \langle \underset{\sim}{u}^{(\alpha)}(\underset{\sim}{k}') \; \underset{\sim}{u}^{(\alpha)}(-\underset{\sim}{k}') \rangle^{(\alpha)} \tag{60}$$

In particular, we have

$$\langle \underset{\sim}{u}'(\underset{\sim}{x}) \; \underset{\sim}{u}'(\underset{\sim}{x}) \rangle' = \int_{-\infty}^{\infty} d\underset{\sim}{k}' \; \chi' \langle \underset{\sim}{u}'(\underset{\sim}{k}') \; \underset{\sim}{u}'(-\underset{\sim}{k}') \rangle' \quad , \tag{61}$$

with $\chi' = \chi^{(1)}$.

In connection with the velocity correction at two instants, we introduce the integrals

$$\eta_{ij}^{(\alpha)}(k) = \int_0^\infty dt' \; \chi^{(\alpha)} \left\langle u_i^{(\alpha)}(t', k) \; u_j^{(\alpha)}(t, -k) \right\rangle^{(\alpha)} \tag{62}$$

with

$$\eta_{ij}^{(\alpha)} = \int_{-\infty}^{\infty} d\underset{\sim}{k} \; \eta_{ij}^{(\alpha)}(k) \; , \tag{63}$$

and

$$\tilde{\eta}^{(\alpha)}(k) = \int_0^\infty dt' \; \chi^{(\alpha)} \left\langle v_i^{(\alpha)}(t', k) \; v_j^{(\alpha)}(t, -k) \right\rangle \tag{64}$$

with

$$\tilde{\eta}_{ij}^{(\alpha)} = \int_{-\infty}^{\infty} d\underset{\sim}{k} \; \tilde{\eta}_{ij}^{(\alpha)}(k) \tag{65}$$

We assume that the turbulent motion is stationary in time within the duration of correlation, and write

$$\left\langle u_i^{(\alpha)}(t', \underset{\sim}{k}) \; u_j^{(\alpha)}(t, -\underset{\sim}{k}) \right\rangle^{(\alpha)} = \left\langle u_i^{(\alpha)}(0, \underset{\sim}{k}) \; u_j^{(\alpha)}(t - t', -\underset{\sim}{k}) \right\rangle^{(\alpha)} , \tag{66}$$

reducing the time integral (62) to

$$\eta_{ij}^{(\alpha)}(k) = \int_0^\infty d\tau \; \chi^{(\alpha)} \left\langle u_i^{(\alpha)}(0, \underset{\sim}{k}) \, u_j^{(\alpha)}(\tau, -\underset{\sim}{k}) \right\rangle^{(\alpha)} , \tag{67}$$

where $\tau = t - t'$. If the turbulent motion of rank α is of a sufficiently small scale, the assumption of isotropy can be applied, giving

$$\eta_{ij}^{(\alpha)} = \eta^{(\alpha)} \delta_{ij} \tag{68}$$

$\eta^{(\alpha)}$ is called an eddy viscosity of the α-th rank.

In view of the averaging rule (57) and of the expected presence of

$$\nabla \langle \underset{\sim}{u}^{(\alpha+1)} \underset{\sim}{u}^{(\alpha+1)} \rangle^{(\alpha+1)} \tag{69}$$

in the equation describing the evolution of $\underset{\sim}{u}^{(\alpha)}$, we deem that $\langle \underset{\sim}{u}^{(\alpha+1)} \underset{\sim}{u}^{(\alpha+1)} \rangle^{\alpha+1}$ has a rank value α. As a consequence, the eddy viscosity $\eta_{ij}^{(\alpha)}$, as obtained by a time integration, which amounts to a smoothing process, will have a rank value α-2 or lower. Thus

$$\langle u_i^{(\alpha)} u_j^{(\alpha)} \rangle^{(\alpha)} \quad \text{has a rank value} \quad \alpha - 1 \tag{70}$$

and

$$\eta_{ij}^{(\alpha)} \quad \text{has a rank value} \quad \leq \alpha - 2 \tag{71}$$

A dynamical equation for the time evolution of a rank $u_i^{(\alpha)}$ may contain terms of other rank values, such as $u_i^{(\beta)}$, with $\beta \neq \alpha$. However, on account of the properties (56) and (57), such terms cannot contribute to calculating statistical quantities, such as $\langle u_i^{(\alpha)} u_j^{(\alpha)} \rangle^{(\alpha)}$, by the dynamical equation. Thus, we have

$$\left\langle u_i^{(\beta)} u_j^{(\alpha)} \right\rangle^{(\alpha)} = u_i^{(\beta)} \left\langle u_j^{(\alpha)} \right\rangle^{(\alpha)} \tag{72}$$

$$= 0, \quad \text{if} \quad \beta < \alpha , \tag{73}$$

where i and j are arbitrary indices. On the other hand, if $\beta > \alpha$, $\underset{\sim}{u}^{(\beta)}$ raises the rank value of $u_i^{(\beta)} u_j^{(\alpha)}$ above α, so that its α-th rank average vanishes

$$\left\langle u_i^{(\beta)} u_j^{(\alpha)} \right\rangle^{(\alpha)} = 0, \quad \text{if} \quad \beta > \alpha . \tag{74}$$

For their usefulness of contributing to statistical quantities, we conclude that all the terms in the dynamical equations of single and repeated cascades must have their rank value conserved. Nevertheless a dynamical equation for the evolution of a rank α may still couple to other ranks, provided they possess the same rank value α.

For the above reason, we can rewrite (70) and (71) as

$$\left\langle \underset{\sim}{u}^{(\alpha)} \underset{\sim}{u}^{(\alpha)} \right\rangle^{(\alpha)} = \left\langle \underset{\sim}{v}^{(\alpha)} \underset{\sim}{v}^{(\alpha)} \right\rangle^{(\alpha)} , \quad \text{rank value} \quad \alpha - 1 \tag{75}$$

$$\eta_{ij}^{(\alpha)} = \tilde{\eta}_{ij}^{(\alpha)} , \quad \text{rank value} \quad \leq \alpha - 2 . \tag{76}$$

IV. DYNAMICAL EQUATIONS IN CASCADE REPRESENTATION

As mentioned in Section III, we will need a repeated cascade for the velocity and only a single cascade for the surface. Therefore, we write their respective dynamical equations as follows:

$$\frac{d u_i^{(\alpha)}(\underset{\sim}{k})}{dt} \equiv \frac{\partial u_i^{(\alpha)}(\underset{\sim}{k})}{\partial t} + \int_{-\infty}^{\infty} dk' \, i k_j' \, V_j^{(\alpha+1)}(\underset{\sim}{k}-\underset{\sim}{k}') \, u_i^{(\alpha)}(\underset{\sim}{k}')$$

$$= - \int_{-\infty}^{\infty} d\underset{\sim}{k}' \, i k_j' \left[u_j^{(\alpha)}(\underset{\sim}{k}-\underset{\sim}{k}') \, V_i^{(\alpha)}(\underset{\sim}{k}') + \left\langle u_j^{(\alpha+1)}(\underset{\sim}{k}-\underset{\sim}{k}') \, u_i^{(\alpha+1)}(\underset{\sim}{k}') \right\rangle^{(\alpha+1)} \right]$$

$$+ \; E_i^{(\alpha)}(\underset{\sim}{k}) - \gamma \, u_i^{(\alpha)}(\underset{\sim}{k}) \qquad (77)$$

$$\frac{D v_i^{(\alpha)}(\underset{\sim}{k})}{Dt} \equiv \frac{\partial v_i^{(\alpha)}(\underset{\sim}{k})}{\partial t} + \int_{-\infty}^{\infty} d\underset{\sim}{k}' \, i k_j' \, u_j(\underset{\sim}{k}-\underset{\sim}{k}') \, v_i^{(\alpha)}(\underset{\sim}{k}')$$

$$= - \int_{-\infty}^{\infty} dk' \, i k_j' \, v_j^{(\alpha)}(\underset{\sim}{k}-\underset{\sim}{k}') \, V_i^{(\alpha)}(\underset{\sim}{k}')$$

$$+ \; \tilde{E}_i^{(\alpha)}(\underset{\sim}{k}) - \gamma \, v_i^{(\alpha)}(\underset{\sim}{k}) \qquad (78)$$

where

$$\tilde{\underset{\sim}{E}}^{(\alpha)} \equiv \underset{\sim}{E}^{(\alpha)} + \cdots + \underset{\sim}{E}^{(N)} \quad , \tag{79}$$

$$\frac{\partial \zeta^{\circ}(\underset{\sim}{k})}{\partial t} + \int_{-\infty}^{\infty} d\underset{\sim}{k}' \, i k_j \, u_j^{\circ}(\underset{\sim}{k}-\underset{\sim}{k}') \, \zeta^{\circ}(\underset{\sim}{k}')$$

$$= -\int_{-\infty}^{\infty} d\underset{\sim}{k}' \, i k_j' \left\langle u_j'(\underset{\sim}{k}-\underset{\sim}{k}') \, \zeta'(\underset{\sim}{k}') \right\rangle' + \xi \, i k^{-1} k_j \, u_j^{\circ}(\underset{\sim}{k}) \tag{80}$$

$$\frac{\partial w^{\circ}(\underset{\sim}{k})}{\partial t} + \int_{-\infty}^{\infty} d\underset{\sim}{k}' \, i k_j \, (k/k') \, u_j^{\circ}(\underset{\sim}{k}-\underset{\sim}{k}') \, w^{\circ}(\underset{\sim}{k}')$$

$$= -\int_{-\infty}^{\infty} d\underset{\sim}{k}' \, i k_j' (k/k') \left\langle u_j'(\underset{\sim}{k}-\underset{\sim}{k}') \, w'(\underset{\sim}{k}') \right\rangle' + \xi \, i c k_j \, u_j^{\circ}(\underset{\sim}{k}) \tag{81}$$

$$\frac{d\zeta'(\underset{\sim}{k})}{dt} \equiv \frac{\partial \zeta'(\underset{\sim}{k})}{\partial t} + \int_{-\infty}^{\infty} d\underset{\sim}{k}' \, i k_j \, u_j(\underset{\sim}{k}-\underset{\sim}{k}') \, \zeta'(\underset{\sim}{k}')$$

$$= -\int_{-\infty}^{\infty} d\underset{\sim}{k}' \, i k_j' \, u_j'(\underset{\sim}{k}-\underset{\sim}{k}') \, \zeta^{\circ}(\underset{\sim}{k}') + \xi \, i k^{-1} k_j \, u_j'(\underset{\sim}{k})$$

$$+ \int_{-\infty}^{\infty} d\underset{\sim}{k}' \, i k_j' \left\langle u_j'(\underset{\sim}{k}-\underset{\sim}{k}') \, \zeta'(\underset{\sim}{k}') \right\rangle' \tag{82}$$

$$\frac{dw'(\underset{\sim}{k})}{dt} \equiv \frac{\partial w'(\underset{\sim}{k})}{\partial t} + \int_{-\infty}^{\infty} d\underset{\sim}{k}' \, i k_j' (k/k') \, u_j(\underset{\sim}{k}-\underset{\sim}{k}') \, w'(\underset{\sim}{k}')$$

$$= -\int_{-\infty}^{\infty} d\underset{\sim}{k}' \, i k_j' (k/k') \, u_j'(\underset{\sim}{k}-\underset{\sim}{k}') \, w^{\circ}(\underset{\sim}{k}') + \xi \, i c k_j \, u_j'(\underset{\sim}{k})$$

$$+ \int_{-\infty}^{\infty} d\underset{\sim}{k}' \, i k_j' (k/k') \left\langle u_j'(\underset{\sim}{k}-\underset{\sim}{k}') \, w'(\underset{\sim}{k}') \right\rangle' \quad . \tag{83}$$

V. LANGEVIN EQUATION AND ONSAGER'S RELATIONS FOR TURBULENT FLUXES

It is to be remarked that D/Dt represents a Lagrangian derivative, i.e. a rate of change following the path of a fluid element. The variable t in D/Dt can be treated as a one-dimensional variable in the Lagrangian representation, notwithstanding its four dimensions in time and three wavenumbers in the Eulerian representation. Under such a circumstance, and in analogy with the Brownian movements of molecules, Eqs. (77), (78), (82) and (83) can be regarded as Langevin equations for turbulent motion, if k is taken to be a parameter. Those Langevin equations will be useful to calculate fluxes and associated transport coefficients of turbulence, as it should be recalled that any transport coefficients are indeed represented by a Lagrangian description in statistical thermodynamics. For this purpose, we make a formal integration of (78), giving

$$v_i^{(\alpha)}(t,\underset{\sim}{k}) = \int_0^t dt' \, h_i^{(\alpha)}(t',\underset{\sim}{k}) \exp[-\gamma(t-t')] + v_i^{(\alpha)}(0,\underset{\sim}{k}) \exp(-\gamma t), \tag{84}$$

where

$$h_i^{(\alpha)}(t',\underset{\sim}{k}) = -\int_{-\infty}^{\infty} d\underset{\sim}{k}' \, i k'_\lambda \, v_\lambda^{(\alpha)}(t',\underset{\sim}{k}-\underset{\sim}{k}') \, V_i^{(\alpha)}(t',\underset{\sim}{k}') + \tilde{E}_i^{(\alpha)}(t,\underset{\sim}{k}) . \tag{85}$$

It is to be noted that a term similar to that in $\langle \cdots \rangle'$ on the right hand side of (82) has not been carried over to (78) and (85), for the reason mentioned in IIIC.

Since a transport property is contributed by a correlation from a rank $\underset{\sim}{v}^{(\alpha)}$ in a quasi-stationary background of rank $\underset{\sim}{V}^{(\alpha)}(t,\underset{\sim}{k})$ in the cascade (43), the upper limit t will belong to the quasi-stationary time scale which is much larger than the duration of that correlation. Therefore that upper limit can be replaced by ∞, and $V_i^{(\alpha)}(t',k')$ can be replaced by $V_i^{(\alpha)}(t,k')$. For the same reason, the initial value will not be correlated with any fluctuation at time t, thus simplifying (84) to

$$v_i^{(\alpha)}(t,\underset{\sim}{k}) = \int_0^\infty dt' \; h_i^{(\alpha)}(t',\underset{\sim}{k}) \exp[-\gamma(t-t')]$$

$$\cong \int_0^\infty d\tau \; h_i^{(\alpha)}(t-\tau,\underset{\sim}{k}) \qquad (86)$$

Here $\tau = t-t'$, and

$$h_i^{(\alpha)}(t-\tau,\underset{\sim}{k}) = -\int_{-\infty}^{\infty} d\underset{\sim}{k}'' \; i k_\ell'' \; v_\ell^{(\alpha)}(t',\underset{\sim}{k}-\underset{\sim}{k}'') \, V_i^{(\alpha)}(t,\underset{\sim}{k}'') + \tilde{E}_i^{(\alpha)}(t,\underset{\sim}{k}) \qquad (87)$$

We have also neglected the friction as being small compared to the eddy mixing process.

The expression (86) for the fluctuation $v_i^{(\alpha)}$ avails itself to formulate a flux

$$\left\langle v_j^{(\alpha)}(t,\underset{\sim}{k}-\underset{\sim}{k}') \, v_i^{(\alpha)}(t,\underset{\sim}{k}') \right\rangle^{(\alpha)} = -\int_{-\infty}^{\infty} d\underset{\sim}{k}'' \; i k_\ell'' \; V_i^{(\alpha)}(t,\underset{\sim}{k}'') \int_0^\infty d\tau \left\langle v_j^{(\alpha)}(t,\underset{\sim}{k}-\underset{\sim}{k}') \, v_\ell^{(\alpha)}(t-\tau,\underset{\sim}{k}'-\underset{\sim}{k}'') \right\rangle^{(\alpha)}$$

$$+ \int_0^\infty d\tau \left\langle v_j^{(\alpha)}(t,\underset{\sim}{k}-\underset{\sim}{k}') \, \tilde{E}_i^{(\alpha)}(t-\tau,\underset{\sim}{k}) \right\rangle^{(\alpha)} \qquad (88)$$

All the terms have the same rank value α-1, except the last term on the right hand side of (85) which has a rank value up to α-2. Such a disparate rank will not contribute to the flux, and therefore will be omitted. When we make use of the property (58) and the definition (64), we reduce the flux to the form

$$\left\langle v_j^{(\alpha)}(t,\underset{\sim}{k}-\underset{\sim}{k}')\, v_i^{(\alpha)}(t,\underset{\sim}{k}')\right\rangle^{(\alpha)} = -\,\tilde{\eta}^{(\alpha)}_{ji}(|\underset{\sim}{k}-\underset{\sim}{k}'|)\, i k_i\, V_i^{(\alpha)}(t,\underset{\sim}{k}) \quad (89)$$

Thus we find the statistical effect of fluctuations of rank $\underset{\sim}{u}'$ upon the evolution of $\underset{\sim}{u}^{\circ}$ to take the form of a flux (89), which is proportional to the background velocity gradient, with the proportionality coefficient given by the eddy viscosity. That a flux is proportional to the gradient of the quantity to be transported seems to fall under the general Onsager relation in thermodynamics of irreversible processes. Therefore, by repeating the method used for the derivation of the momentum flux, we obtain the similar relations for the surface fluxes

$$\left\langle u_j'(t,\underset{\sim}{k}-\underset{\sim}{k}')\,\zeta'(t,\underset{\sim}{k}')\right\rangle' = -\,\eta'_{ji}(|\underset{\sim}{k}-\underset{\sim}{k}'|)\, i k_i\, \zeta^{\circ}(t,\underset{\sim}{k}) \quad (90)$$

$$\left\langle u_j'(t,\underset{\sim}{k}-\underset{\sim}{k}')\,w'(t,\underset{\sim}{k}')\right\rangle' = -\,\eta'_{ji}(|\underset{\sim}{k}-\underset{\sim}{k}'|)\, i k_i (k'/k)\, w^{\circ}(t,\underset{\sim}{k}) \,. \quad (91)$$

The relations (89), (90) and (91) will be called the Onsager relations of turbulence. We shall not have the need of fluxes of higher ranks arizing from $\zeta^{(\alpha)}$ and $w^{(\alpha)}$, and therefore they are not written explicitly here.

VI. TRANSPORTS OF MOMENTUM AND ENERGY

By relying upon the Onsager relation (89), we can transform the equation (77) for the momentum transport into the form

$$\left[\frac{d}{dt} + \omega'(k)\right] u_i^\circ(t,\underset{\sim}{k}) = E_i^\circ(t,\underset{\sim}{k}) - \gamma\, u_i^\circ(t,\underset{\sim}{k}) , \tag{92}$$

where

$$\omega'(k) = \int_{-\infty}^{\infty} d\underset{\sim}{k}'\; k_j' k_s' \eta_{js}'\left(|\underset{\sim}{k}-\underset{\sim}{k}'|\right) - k_j k_s \eta_{js}' \tag{93}$$

is a relaxation frequency.

We can assume an isotropic turbulence for the small eddies contributing η_{js}', and write

$$\omega'(k) = k^2 \eta' \tag{94}$$

If we treat (92) as a Langevin equation, and integrate, we find

$$u_i^\circ(t,k) = \int_0^\infty d\tau\; E_i^\circ(t-\tau,k) \exp(-\omega'\tau) . \tag{95}$$

Here the initial value is omitted, because it does not contribute in any correlation. In addition we have neglected the friction γ as compared to the eddy relaxation frequency ω'.

In a similar way, we reduce the equations (80) and (81) for the transport of ζ° and w° into the following:

$$\left[\frac{d}{dt} + \omega'(k)\right] \zeta^\circ(t,\underset{\sim}{k}) = \xi\, i\, k^{-1} \underset{\sim}{k} \cdot \underset{\sim}{u}^\circ(t,\underset{\sim}{k}) \tag{96}$$

$$\left[\frac{d}{dt} + \omega'(k)\right] w^\circ(t,\underset{\sim}{k}) = \xi\, i c\, \underset{\sim}{k} \cdot \underset{\sim}{u}^\circ(t,\underset{\sim}{k}) \ , \tag{97}$$

or, in terms of E_i°,

$$\left[\frac{d}{dt} + \omega'(k)\right] E_i^\circ(t,\underset{\sim}{k}) = \xi\, c^2 k_i\, \underset{\sim}{k} \cdot \underset{\sim}{u}^\circ(t,\underset{\sim}{k}) \ . \tag{98}$$

The equations (92), (96) and (97) for the transpors of $\underset{\sim}{u}^\circ$, ζ° and w° take now a form considerably simpler than the original form in Eqs. (77), (80) and (81). The simpler form casts its nonlinear transfer into the relaxation frequency ω'.

Upon multiplying (92), (96) and (97) by $u_i^\circ(-k)$, $\zeta^\circ(-k)$ and $w^\circ(-k)$, respectively, and taking an average, we find the equations of energy in Fourier form:

$$\left(\frac{\partial}{\partial t} + \omega'\right)\left\langle u_i^\circ(\underset{\sim}{k})\, u_i^\circ(-\underset{\sim}{k})\right\rangle^\circ = E_i^\circ(\underset{\sim}{k}) E_i^\circ(-\underset{\sim}{k}) - \gamma\, u_i^\circ(\underset{\sim}{k})\, u_i^\circ(-\underset{\sim}{k}) + (\underset{\sim}{k} \to -\underset{\sim}{k}) \tag{99}$$

$$\left(\frac{\partial}{\partial t} + \omega'\right)\left\langle \zeta^\circ(\underset{\sim}{k})\, \zeta^\circ(-\underset{\sim}{k})\right\rangle^\circ = \xi\, (gk)^{-1} \left\langle E_i^\circ(-\underset{\sim}{k})\, u_i^\circ(\underset{\sim}{k})\right\rangle^\circ + (\underset{\sim}{k} \to -\underset{\sim}{k}) \tag{100}$$

$$\left(\frac{\partial}{\partial t} + \omega'\right)\left\langle w^\circ(\underset{\sim}{k})\, w^\circ(-\underset{\sim}{k})\right\rangle^\circ = \xi \left\langle E_i^\circ(-\underset{\sim}{k})\, u_i^\circ(\underset{\sim}{k})\right\rangle^\circ + (\underset{\sim}{k} \to -\underset{\sim}{k}) \tag{101}$$

In the above equations (99), (100) and (101), the convection terms do not contribute to the energy evolution in homogeneous turbulence, and are therefore omitted. The complex conjugate part is represented by $(\underset{\sim}{k} \to -\underset{\sim}{k})$, as obtained from replacing $\underset{\sim}{k}$ by $-\underset{\sim}{k}$.

When we multiply (99), (100) and (101) by a scaling factor χ°, and integrate with respect to $\underset{\sim}{k}$, as prescribed by (60), we obtain the equations of energy balance

$$\frac{\partial}{\partial t}\int_0^k dk' F(k') \equiv \frac{1}{2}\frac{\partial}{\partial t}\int_{-\infty}^{\infty} d\underset{\sim}{k}' \chi^\circ \langle u_i^\circ(\underset{\sim}{k}')\, u_i^\circ(-\underset{\sim}{k}')\rangle^\circ = \Gamma^\circ - T^\circ - \nu J^\circ - \psi^\circ \tag{102}$$

$$\frac{\partial}{\partial t}\int_0^k dk' H(k') \equiv \frac{1}{2}\frac{\partial}{\partial t}\int_{-\infty}^{\infty} d\underset{\sim}{k}' \chi^\circ \langle \zeta^\circ(\underset{\sim}{k}')\, \zeta^\circ(-\underset{\sim}{k}')\rangle^\circ = \xi\Gamma_\zeta^\circ - T_\zeta^\circ - \lambda J_\zeta^\circ \tag{103}$$

$$\frac{\partial}{\partial t}\int_0^k dk' G(k') \equiv \frac{1}{2}\frac{\partial}{\partial t}\int_{-\infty}^{\infty} d\underset{\sim}{k}' \chi^\circ \langle w^\circ(\underset{\sim}{k}')\, w^\circ(-\underset{\sim}{k}')\rangle^\circ = \xi\Gamma^\circ - T_w^\circ - \lambda J_w^\circ \;, \tag{104}$$

with the following transport functions:

(a) for the kinetic energy

$$\begin{aligned}
T^\circ &= \eta' J^\circ \\
J^\circ &= 2\int_0^k dk'\, k'^2\, F(k') \\
&= \int_{-\infty}^{\infty} d\underset{\sim}{k}'\, k'^2\, \chi^\circ \langle \underset{\sim}{u}^\circ(t,\underset{\sim}{k})\cdot \underset{\sim}{u}^\circ(t,-\underset{\sim}{k})\rangle^\circ \\
\psi^\circ &= 2\gamma\int_0^k dk'\, F(k') \\
&= \gamma\int_{-\infty}^{\infty} dk'\, \chi^\circ \langle \underset{\sim}{u}^\circ(t,\underset{\sim}{k})\cdot \underset{\sim}{u}^\circ(t,-\underset{\sim}{k})\rangle^\circ
\end{aligned}$$

$$\Gamma^\circ = \int_{-\infty}^{\infty} dk'\, \Gamma^\circ(k') \;, \qquad \Gamma^\circ(k) = \chi^\circ \langle \underset{\sim}{E}^\circ(t,\underset{\sim}{k})\cdot \underset{\sim}{u}^\circ(t,-\underset{\sim}{k})\rangle^\circ \tag{105}$$

(b) for the surface

$$T_\zeta^\circ = \eta' J_\zeta^\circ$$

$$J_\zeta^\circ \equiv 2\int_0^k dk'\, k'^2 H(k') = \int_{-\infty}^{\infty} dk'\, k'^2 \chi^\circ \langle \zeta^\circ(t,k)\zeta^\circ(t,-k)\rangle^\circ$$

$$\Gamma_\zeta^\circ \equiv \int_{-\infty}^{\infty} dk'\, \Gamma_\zeta^\circ(k'), \quad \Gamma_\zeta^\circ(k) = (gk)^{-1}\chi^\circ \langle \underset{\sim}{E}^\circ(t,\underset{\sim}{k})\cdot \underset{\sim}{u}^\circ(t,-\underset{\sim}{k})\rangle^\circ \tag{106}$$

(c) for the potential energy

$$T_w^\circ = \eta' J_w^\circ$$

$$J_w^\circ \equiv 2\int_0^k dk'\, k'^2 G(k') = \int_{-\infty}^{\infty} d\underset{\sim}{k}'\, k'^2 \chi^\circ \langle w^\circ(t,\underset{\sim}{k}) w^\circ(t,-\underset{\sim}{k})\rangle^\circ \tag{107}$$

In the functions (105), (106) and (107), we have not written out the complex conjugate part explicitly, as we did in (99), (100) and (101), because the integration with respect to $d\underset{\sim}{k}$ in the interval $(-\infty, \infty)$ rules out this necessity. The functions T°, T_ζ° and T_W° are called transfer functions, they govern the cascade transfer of energy across each individual spectrum. The terms νJ°, λJ_ζ° and λJ_W° are dissipation functions, proportional to the vorticity functions J°, J_ζ° and J_W°, with the molecular viscosity ν and diffusivity λ as proportionality coefficients. Although the dynamical equations (102), (103) and (104) were established without consideration of dissipations, the dissipations are now added to keep a conservation of energy, as an often allowed practice in hydrodynamical problems. Finally Γ° and Γ°_{ζ} are called gravitational exchange functions, and they govern the exchange of energy between two portions of separte spectra which are subject to the gravitational pull.

VII. GRAVITATIONAL EXCHANGE

The gravitational exchange function (105) involves the coupling $<\underset{\sim}{E}^\circ(t,\underset{\sim}{k})\cdot\underset{\sim}{u}^\circ(t,-\underset{\sim}{k})>^\circ$, which can be calculated by means of (92), and gives

$$\left\langle \underset{\sim}{E}^\circ(t,k)\cdot\underset{\sim}{u}^\circ(t,-\underset{\sim}{k})\right\rangle^\circ = \int_0^\infty d\tau \left\langle \underset{\sim}{E}^\circ(t,\underset{\sim}{k})\cdot\underset{\sim}{E}^\circ(t-\tau,-\underset{\sim}{k})\right\rangle^\circ \exp(-\omega'\tau) . \tag{108}$$

The correlation under the integrand in (108) has its development governed by Eq. (98), which is

$$\left(\frac{d}{dt}+\omega'\right)\left\langle E_i^\circ(t,\underset{\sim}{k})\,E_i^\circ(t',-\underset{\sim}{k})\right\rangle^\circ = \xi c^2 k_i k_j \left\langle u_j^\circ(t,\underset{\sim}{k})\,E_i^\circ(t',-\underset{\sim}{k})\right\rangle^\circ . \tag{109}$$

The isotropic form is valid for the present isotropic turbulence

$$\left(\frac{d}{d\tau}+\omega'\right)\left\langle E_i^\circ(t,\underset{\sim}{k})\,E_i^\circ(t-\tau,-\underset{\sim}{k})\right\rangle^\circ = \xi\ \theta^2\omega'^2\left\langle u_i^\circ(t,\underset{\sim}{k})\,E_i^\circ(t-\tau,-\underset{\sim}{k})\right\rangle^\circ , \tag{110}$$

and can be integrated to give

$$\begin{aligned}\left\langle \underset{\sim}{E}^\circ(t,\underset{\sim}{k})\cdot\underset{\sim}{E}^\circ(t-\tau,-\underset{\sim}{k})\right\rangle^\circ &= \left\langle \underset{\sim}{E}^\circ(t,\underset{\sim}{k})\cdot\underset{\sim}{E}^\circ(t,-\underset{\sim}{k})\right\rangle^\circ \exp(-\omega'\tau) \\ &+ \xi\,\theta^2\omega'^2\int_0^\tau d\tau' \left\langle \underset{\sim}{u}^\circ(t,\underset{\sim}{k})\cdot\underset{\sim}{E}^\circ(t-\tau',-\underset{\sim}{k})\right\rangle^\circ \exp[-\omega'(\tau-\tau')] ,\end{aligned} \tag{111}$$

where

$$\theta \equiv kc/2\omega' \ll 1 \tag{112}$$

is the ratio of the frequency of oscillation of a sinusoidal gravity wave to the turbulent frequency. Since the generating gravity wave has a frequency lying in the production subrange, and the turbulent frequency is in the inertia or higher frequency subranges, that ratio is a small quantity.

When we substitute the expression (111) for the $\underset{\sim}{E}^{\circ}$-correlation into (108), we find

$$\langle \underset{\sim}{E}^{\circ}(t,\underset{\sim}{k})\cdot\underset{\sim}{u}^{\circ}(t,-\underset{\sim}{k})\rangle^{\circ} = \frac{1}{2\omega'}\langle \underset{\sim}{E}^{\circ}(t,\underset{\sim}{k})\cdot\underset{\sim}{E}^{\circ}(t,-\underset{\sim}{k})\rangle^{\circ}$$

$$+\xi\,\theta^{2}\omega'^{2}\int_{0}^{\infty}d\tau\int_{0}^{\tau}d\tau'\,\langle \underset{\sim}{u}^{\circ}(t,\underset{\sim}{k})\cdot\underset{\sim}{E}^{\circ}(t-\tau',-\underset{\sim}{k})\rangle^{\circ}\exp[-\omega'(2\tau-\tau')]\ . \tag{113}$$

The order of the double integration can be inverted to read

$$\int_{0}^{\infty}d\tau\int_{0}^{\tau}d\tau'\,\exp(-2\omega'\tau)\ldots = \int_{0}^{\infty}d\tau'\int_{\tau'}^{\infty}d\tau\,\exp(-2\omega'\tau)\ldots$$

$$= \frac{1}{2\omega'}\int_{0}^{\infty}d\tau'\,\exp(-2\omega'\tau)\ldots\quad , \tag{114}$$

and reduce (113) to

$$\langle \underset{\sim}{E}^{\circ}(t,\underset{\sim}{k})\cdot\underset{\sim}{u}^{\circ}(t,-\underset{\sim}{k})\rangle^{\circ} = \frac{1}{2\omega'}\langle \underset{\sim}{E}^{\circ}(t,\underset{\sim}{k})\cdot\underset{\sim}{E}^{\circ}(t,-\underset{\sim}{k})\rangle^{\circ}$$

$$+\ \frac{1}{2}\xi\,\theta^{2}\omega'\int_{0}^{\infty}d\tau'\,\langle \underset{\sim}{u}^{\circ}(t,\underset{\sim}{k})\cdot\underset{\sim}{E}^{\circ}(t-\tau',-\underset{\sim}{k})\rangle^{\circ}\ . \tag{115}$$

It is to be remarked that the integral term on the right hand is of the order of $\frac{1}{2}\theta^{2}\langle\underset{\sim}{E}^{\circ}(t,\underset{\sim}{k})\cdot\underset{\sim}{u}^{\circ}(t,-\underset{\sim}{k})\rangle^{\circ}$, and is a factor θ^{2} smaller than the term on the left hand side. When such a term is neglected, we simplify (115) to

$$\langle \underset{\sim}{E}^{\circ}(t,\underset{\sim}{k}) \cdot \underset{\sim}{u}^{\circ}(t,-\underset{\sim}{k}) \rangle^{\circ} = \frac{1}{2\omega'} \langle \underset{\sim}{E}^{\circ}(t,\underset{\sim}{k}) \cdot \underset{\sim}{E}^{\circ}(t,-\underset{\sim}{k}) \rangle^{\circ} \quad . \tag{116}$$

It follows the expressions for the gravitational exchanges written in terms of the spectral function H(k):

$$\Gamma^{\circ} = \int_{-\infty}^{\infty} d\underset{\sim}{k}' \,(2\omega')^{-1} \chi^{\circ} \langle \underset{\sim}{E}^{\circ}(t,\underset{\sim}{k}') \cdot \underset{\sim}{E}^{\circ}(t,-\underset{\sim}{k}') \rangle^{\circ} = \int_{0}^{k} dk' \, \frac{[k' c(k')]^{4} H(k')}{\omega'(k')} \tag{117}$$

$$\Gamma_{c}^{\circ} \equiv \int_{-\infty}^{\infty} d\underset{\sim}{k}' \left[2\,\omega'(k')\, g^{*}(k')\, k' \right]^{-1} \chi^{\circ} \langle \underset{\sim}{E}^{\circ}(t,\underset{\sim}{k}') \cdot \underset{\sim}{E}^{\circ}(t,-\underset{\sim}{k}') \rangle^{\circ}$$

$$= \int_{0}^{k} dk' \, \frac{[k' c(k')]^{2} H(k')}{\omega'(k')} \quad . \tag{118}$$

VIII. EDDY VISCOSITY AND MEMORY CHAIN

VIIIA. Structure of Eddy Viscosity

A turbulent motion, when decomposed into a macroscopic background and a random fluctuation, finds an interplay between the two parts in such a way that the statistical effect of the random fluctuations upon the macroscopic background appears as a turbulent stress, expressed by the Onsager relation (89) as derived by the Langevin equation (78) of fluctuations. That relation enters the eddy viscosity $\eta^{(\alpha)}$ as the proportionality coefficient. Now in order to substantiate $\eta^{(\alpha)}$, it is necessary to extend a correlation $\langle \underset{\sim}{u}^{(\alpha)}(o)\cdot\underset{\sim}{u}^{(\alpha)}(\tau)\rangle^{\alpha}$ over a certain time period commensurate with the relaxation frequency $\omega^{(\alpha+1)}$ of smaller scales. We shall study the development of that correlation from the momentum equation (77), which is rewritten as

$$\left(\frac{d}{dt}+\omega^{(\alpha+1)}\right)u_i^{(\alpha)}(t,\underset{\sim}{k})$$

$$= -\int_{-\infty}^{\infty} d\underset{\sim}{k}'\, i k_j'\, u_j^{(\alpha)}(t,\underset{\sim}{k}-\underset{\sim}{k}')\, V^{(\alpha)}(t,\underset{\sim}{k}') + E_i^{(\alpha)}(t,\underset{\sim}{k}) \quad , \tag{119}$$

after substituting for the stress from (89). Here

$$\omega^{(\alpha+1)} = c_1 k^2 \eta^{(\alpha+1)} \tag{120}$$

is a relaxation frequency of rank $\alpha+1$.

Upon multiplying (119) by $u_i^{(\alpha)}(t',-\underset{\sim}{k})$ and averaging, we find

$$\left(\frac{d}{dt}+\omega^{(\alpha+1)}\right)\left\langle u_i^{(\alpha)}(t,\underset{\sim}{k})\, u_i^{(\alpha)}(t',-\underset{\sim}{k})\right\rangle^{(\alpha)}$$

$$= -\int_{-\infty}^{\infty} d\underset{\sim}{k}'\, i k_j' \left\langle u_j^{(\alpha)}(t,\underset{\sim}{k}-\underset{\sim}{k}')\, u_i^{(\alpha)}(t',-\underset{\sim}{k})\right\rangle^{(\alpha)} V_i^{(\alpha)}(t,\underset{\sim}{k}')$$

$$+ \left\langle E_i^{(\alpha)}(t,\underset{\sim}{k})\, u_i^{(\alpha)}(t',-\underset{\sim}{k})\right\rangle^{(\alpha)} . \tag{121}$$

The second term on the right hand side vanishes on account of (70). After an integration, we have

$$\left\langle u_i^{(\alpha)}(t,\underset{\sim}{k})\, u_i^{(\alpha)}(t-\tau,-\underset{\sim}{k})\right\rangle^{(\alpha)} = \left\langle u_i^{(\alpha)}(t,\underset{\sim}{k})\, u_i^{(\alpha)}(t,-\underset{\sim}{k})\right\rangle^{(\alpha)} \exp\left(-\omega^{(\alpha+1)}\tau\right)$$

$$+ \int_0^{\tau} d\tau' \left\langle E_i^{(\alpha)}(t,\underset{\sim}{k})\, u_i^{(\alpha)}(t-\tau',-\underset{\sim}{k})\right\rangle^{(\alpha)} . \tag{122}$$

A second integration gives

$$\int_0^{\infty} d\tau \left\langle u_i^{(\alpha)}(t,\underset{\sim}{k})\, u_i^{(\alpha)}(t-\tau,-\underset{\sim}{k})\right\rangle^{(\alpha)} = \frac{1}{\omega^{(\alpha+1)}}\left\langle u_i^{(\alpha)}(t,\underset{\sim}{k})\, u_i^{(\alpha)}(t,-\underset{\sim}{k})\right\rangle^{(\alpha)}$$

$$+ \frac{1}{\omega^{(\alpha+1)}}\int_0^{\infty} d\tau \left\langle E_i^{(\alpha)}(t,\underset{\sim}{k})\, u_i^{(\alpha)}(t-\tau,-\underset{\sim}{k})\right\rangle^{(\alpha)}. \tag{123}$$

A double time integration involved in the above and was handled like in (114).

On the right hand side of (123), the last term can be computed from writing an equation describing the evolution of E_i^{α}, similar to (98). As a result, we reduce that term to the same order and form as the left hand side of (123), but at a factor

θ^2 smaller. The details of the calculations are similar to those leading to (116), and will be omitted. Thus, by neglecting such a term, we find

$$\eta^{(\alpha)} = \int_{-\infty}^{\infty} d\underset{\sim}{k} \, \frac{\langle u_i^{(\alpha)}(t, \underset{\sim}{k}) \, u_i^{(\alpha)}(t, -\underset{\sim}{k}) \rangle^{(\alpha)}}{2\,\omega^{(\alpha+1)}} \quad , \tag{124}$$

and

$$\tilde{\eta}^{(\alpha)} = \int_{-\infty}^{\infty} d\underset{\sim}{k} \, \frac{\langle v_i^{(\alpha)}(t, \underset{\sim}{k}) \, v_i^{(\alpha)}(t, -\underset{\sim}{k}) \rangle^{(\alpha)}}{2\,\omega^{(\alpha+1)}} \tag{125}$$

with

$$\tilde{\omega}^{(\alpha+1)}(k) = k^2 \, \tilde{\eta}^{(\alpha+1)}$$

where $\eta^{(\alpha)}$ and $\tilde{\eta}^{(\alpha)}$ have been defined by (63) and (65).

The formula for the eddy viscosity $\eta^{(o)}$, obtained by putting $\alpha = o$ in (124), is similar to the gravitational exchange function Γ° in (117), except with a different controlling energy.

VIIIB. Memory Chain

We note that the transport functions T°, T°_{ζ}, T°_{W}, Γ° and Γ°_{ζ}, as derived from the single cascade method, depend on the eddy viscosity of first rank $\eta' \equiv \tilde{\eta}^{(1)}$. However, the approach to equilibrium of $\tilde{\eta}^{(1)}$ entails a memory chain, as exemplified by the formula (125), and rewritten explicitly in terms of a spectral distribution of kinetic energy $F(k)$, defined by (105),

$$\tilde{\eta}^{(1)}(x/k) = \int_k^\infty dk^{(1)} \frac{F(k^{(1)})}{k^{(1)2}\,\tilde{\eta}^{(2)}(x/k^{(1)})}$$

$$\tilde{\eta}^{(2)}(x/k^{(1)}) = \int_{k^{(1)}}^\infty dk^{(2)} \frac{F(k^{(2)})}{k^{(2)2}\,\tilde{\eta}^{(3)}(x/k^{(2)})} ,$$

$$\vdots$$

and the chain continues. Here $\eta^{(\alpha)}(x/k^{(\alpha-1)})$ is an eddy viscosity of the α-th rank in the x-space. The lower limiting wave number $k^{(\alpha-1)}$, according to the definition (50), is written into the argument.

In Ref. 5 we have discussed a method of cutoff of the memory chain by molecular damping, in which the eddy relaxation frequency will be taken over by a frictional frequency at sufficiently high rank of the chain. The result helps in determining the cutoff spectrum near the tail of the dissipation subrange. Since this is not of special interest in the present work, we make an inviscid approximation, by writing

$$\tilde{\eta}^{(1)} = \eta'(x/k) \tag{127}$$

$$\tilde{\eta}^{(2)} \cong \eta'(x/k^{(1)}) , \tag{128}$$

with the eddy viscosity (128) set at a higher limiting wave number than the eddy viscosity (127), simplifying (126) to

$$\eta' = \int_k^\infty dk' \frac{F(k')}{k'^2\,\eta'(x/k')} \tag{129}$$

The integral equation (129) yields the following solution

$$\eta' = \left[2 \int_{k}^{\infty} dk' \, k'^{-2} \, F(k') \right]^{\frac{1}{2}} . \qquad (130)$$

IX. UNIVERSAL RANGE OF SPECTRUM

The universal range holds at sufficiently large wave numbers, such that

$$\frac{\partial}{\partial t}\int_0^k dk'\, F(k') , \qquad \frac{\partial}{\partial t}\int_0^k dk'\, G(k') , \qquad \frac{\partial}{\partial t}\int_0^k dk'\, H(k')$$

become independent of k. Noting that

$$T^\circ(k=\infty) = 0 , \qquad T_\zeta^\circ(k=\infty) = 0 ,$$

we can rewrite Eqs. (102), (103) and (104) in new forms for $k = \infty$, and subtract the new forms from the original equations, yielding

$$-\Gamma^\circ + T^\circ + \nu J^\circ + \Psi^\circ = -\Gamma + \varepsilon + \Psi \tag{131}$$

$$-\xi\Gamma_\zeta^\circ + T_\zeta^\circ + \lambda J_\zeta^\circ = -\xi\Gamma_\zeta + \varepsilon_\zeta \tag{132}$$

$$-\xi\Gamma^\circ + T_w^\circ + \lambda J_w^\circ = -\xi\Gamma + \varepsilon_w , \tag{133}$$

with the notations

$$\begin{aligned}
\Gamma &= \Gamma^\circ(k=\infty) , & \Gamma_\zeta &= \Gamma_\zeta^\circ(k=\infty) , & \Psi &= \Psi^\circ(k=\infty) , \\
J &= J^\circ(k=\infty) , & J_\zeta &= J_\zeta^\circ(k=\infty) , & J_w &= J_w^\circ(k=\infty) , \\
\varepsilon &= \nu J , & \varepsilon_\zeta &= \lambda J_\zeta , & \varepsilon_w &= \lambda J_w .
\end{aligned} \tag{134}$$

IX. UNIVERSAL RANGE OF SPECTRUM

The universal range holds at sufficiently large wave numbers, such that

$$\frac{\partial}{\partial t}\int_0^k dk'\, F(k') , \qquad \frac{\partial}{\partial t}\int_0^k dk'\, G(k') , \qquad \frac{\partial}{\partial t}\int_0^k dk'\, H(k')$$

become independent of k. Noting that

$$T^{\circ}(k=\infty) = 0 , \qquad T^{\circ}_{\zeta}(k=\infty) = 0 ,$$

we can rewrite Eqs. (102), (103) and (104) in new forms for $k = \infty$, and subtract the new forms from the original equations, yielding

$$-\Gamma^{\circ} + T^{\circ} + \nu J^{\circ} + \Psi^{\circ} = -\Gamma + \varepsilon + \Psi \tag{131}$$

$$-\xi\, \Gamma^{\circ}_{\zeta} + T^{\circ}_{\zeta} + \lambda J^{\circ}_{\zeta} = -\xi\, \Gamma_{\zeta} + \varepsilon_{\zeta} \tag{132}$$

$$-\xi\, \Gamma^{\circ} + T^{\circ}_{w} + \lambda J^{\circ}_{w} = -\xi\, \Gamma + \varepsilon_{w} , \tag{133}$$

with the notations

$$\begin{aligned} &\Gamma = \Gamma^{\circ}(k=\infty) , && \Gamma_{\zeta} = \Gamma^{\circ}_{\zeta}(k=\infty) , && \Psi = \Psi^{\circ}(k=\infty) , \\ &J = J^{\circ}(k=\infty) , && J_{\zeta} = J^{\circ}_{\zeta}(k=\infty) , && J_{w} = J^{\circ}_{w}(k=\infty) , \\ &\varepsilon = \nu J , && \varepsilon_{\zeta} = \lambda J_{\zeta} , && \varepsilon_{w} = \lambda J_{w} . \end{aligned} \tag{134}$$

In the present quasi-stationary process, the coupled integral equations do not have the time t as an explicit variable, with the understanding that the spectral distributions F, G and H may vary slowly with time through the physical parameters ε, ε_ζ and ε_w.

X. INERTIA AND DISSIPATION SUBRANGES FOR STABLE AND UNSTABLE CONFIGURATIONS (ξ = -1 and +1)

We refer the configuration of a heavy liquid above a lighter one as unstable (ξ = 1), and the reverse configuration, e.g. sea surface, as stable (ξ = -1). Their distinction lies in the roles of the gravitational pull: in the unstable configuration, the gravitational pull becomes an energy source for both the surface elevation and the velocity which endows it, while in the stable case, the gravitational pull serves to produce the kinetic energy at the expense of the potential energy, so that the two energies must balance. The gravitational pull is represented by the exchange functions $\Gamma-\Gamma^{\circ}$ and $\Gamma_{\zeta}-\Gamma_{\zeta}^{\circ}$ in Eqs. (131) and (132). Since they operate only at small wave number, they become absent in the inertia and dissipation subranges, so that the spectral laws will be valid for both stable and unstable configurations.

Under those circumstances, the equations (131) and (132) governing the spectral distributions reduce to

$$(\nu + \eta')J^{\circ} + 2\gamma\int_0^k dk' F(k') = \varepsilon + \psi \tag{134}$$

$$(\lambda + \eta')J_{\zeta}^{\circ} = \varepsilon_{\zeta} \tag{135}$$

The first of the system (134) and (135) becomes decoupled from the second, and can be solved independently. The flow of energy between transfer and friction can be best described by a differential form

$$\frac{d\eta'}{dk} J + \nu \frac{dJ^{\circ}}{dk} + 2\gamma F = 0 \quad , \tag{136}$$

obtained by approximating

$$J^{\circ} \cong J \qquad \text{and} \qquad \nu + \eta' \cong \nu \ .$$

The solution of (136) is

$$F = \frac{1}{2} J^{2} (\gamma + 2\nu k^{2})(\gamma + \nu k^{2})^{-3} k^{-3} \tag{137}$$

It follows, from (137), the solution of (135),

$$H = \frac{1}{2} \frac{J_{\zeta} J}{\lambda} k^{-5} \left[1 + \frac{1}{2} \frac{J}{\lambda} \frac{k^{-2}}{\nu + \gamma k^{2}} \right]^{-2} \frac{\gamma + 2\nu k^{2}}{(\gamma + \nu k^{2})^{2}} \tag{138}$$

The general solutions (137) and (138) cover subranges including the three parameters γ, ν and λ. We shall consider the following three subranges:

(a) Inertia subrange with friction (ν = o, λ = o)

By writing ν = o and λ = o, we reduce (137) and (138) to

$$F = \frac{1}{2} (J/\gamma)^{2} k^{-3} \ , \tag{139}$$

$$H = 2 (\varepsilon_{\zeta} \gamma / J) k^{-1} \ . \tag{140}$$

(b) Dissipation subrange with surface friction (ν = o)

When the spectrum $F(k)$ falls by friction, as

$$F = \frac{1}{2}(J/\gamma)^2 k^{-3}, \tag{142}$$

the surface spectrum is dissipated by its molecular diffusivity λ and becomes

$$H = \frac{1}{2}(J_\zeta J/\gamma\lambda) k^{-5}. \tag{143}$$

(c) Dissipation subrange without friction (γ = o)

By writing γ = o, we reduce the general solutions (137) and (138) to

$$F = (J/\nu)^2 k^{-7} \tag{144}$$

$$H = (J_\zeta J/\lambda\nu) k^{-7}, \tag{145}$$

in agreement with the Heisenberg law of (1).

(d) Inertia subrange without friction

The inertia subrange without friction reduces further Eqs. (135) and (136) to a decoupled system

$$\eta' J^\circ = \varepsilon \tag{146}$$

$$\eta' J_\zeta^\circ = \varepsilon_\zeta, \tag{147}$$

which gives the solutions

$$F = 0.83\, \varepsilon^{2/3} k^{-5/3} \tag{148}$$

$$H = 0.85\,\varepsilon_\zeta\,\varepsilon^{-1/3} k^{-5/3} \tag{149}$$

in agreement with the Kolmogoroff law of (1). It is to remarked that free medium and a moving surface have two different numerical coefficients in their inertia subrange.

XI. GENERATION OF TURBULENCE BY GRAVITATIONAL INSTABILITY ($\xi = 1$)

In view of the unstable configuration of a heavy liquid above a lighter one (i.e. $\xi = 1$), a turbulent interface may be generated by the gravitational instability. Unlike the stable sea surface, where the effects of the gravitational pull balances themselves in the kinetic energy and the potential energy, the present unstable configuration relies upon the transfer function for transferring energy across a spectrum, and upon the gravity exchange function for feeding energy separately into each spectrum. Therefore, in the production subrange where the above mechanism is in effect, the molecular dissipations can be neglected, reducing the equation of energy balance to

$$-\int_0^k dk' \, \frac{[c(k')]^4 k'^2 H(k')}{\eta'(\underset{\sim}{x}/k')} + \eta' J^\circ + \psi^\circ = \Gamma + \varepsilon + \psi \tag{150}$$

$$-\int_0^k dk' \, \frac{[c(k')]^2 H(k')}{\eta'(\underset{\sim}{x}/k')} + \eta' J^\circ_\zeta = \Gamma_\zeta + \varepsilon_\zeta \tag{151}$$

The equation of potential energy is not useful here, because there will be no balance between the kinetic energy and the potential energy.

The flow of energy from the gravitational instability into the wave coupling by inertia can be more explicitly demonstrated from a differential form of (150) and (151),

$$-\frac{k^2 c^4 H}{\eta'} + \eta' 2k^2 F + J^\circ \frac{d\eta'}{dk} + \gamma F = 0 \tag{152}$$

$$- \frac{c^2 H}{\eta'} + \eta' 2 k^2 H + J_\zeta^\circ \frac{d\eta'}{dk} = 0 \quad . \tag{153}$$

The gravitational pull in the unstable configuration provides a natural source of energy, for maintaining both spectra in their respective subranges of production, and for keeping them from being disintegrated into dissipation. Under those circumstances, the vorticity functions J^0, and J_ζ^0 controlling the eddy dissipations, can be neglected, reducing (152) and (153) to

$$-c^4 k^2 H + 2\eta'^2 k^2 F + \gamma \eta' F = 0 \tag{154}$$

and

$$-c^2 H + 2\eta'^2 k^2 H = 0 \; , \tag{155}$$

where η' is given by (130).

We find the solutions

$$F = A_1 g \, k^{-2} \left(1 + \frac{T}{3\rho g} k^2\right) \tag{156}$$

$$H = A_1 k^{-3} \frac{1 + (T/3\rho g) k^2}{1 + (T/\rho g) k^2}$$

$$\times \left\{ 1 + (k_\gamma / k)^{\frac{1}{2}} \left[1 + (T/\rho g) k^2\right]^{-\frac{1}{2}} \right\} , \tag{157}$$

with

$$A_1 = \frac{3}{4} \tag{158}$$

$$B_1 = A_1/\sqrt{2} \cong 0.53 \ , \tag{159}$$

and

$$k_\gamma = C_1 \gamma^2/g \ , \quad C_1 = (B_1/A_1)^2 \cong 0.5 \tag{160}$$

is a frictional transition wave number separating the two regimes:

(a) Non-frictional, $k \gg k_\gamma$, and $T = 0$

$$H = A_1 k^{-3} \ , \tag{161}$$

(b) Frictional, $k \ll k_\gamma$, and $T = 0$

$$H = A_1 k_\gamma^{\frac{1}{2}} k^{-3.5} \tag{162}$$

XII. SEA SURFACE TURBULENCE CONTROLLED BY GRAVITATIONAL PULL

In a stable configuration ($\varepsilon = -1$) like the sea surface, the effect of gravity is to pull down the surface elevation, and, during this course of action, to disperse the surface liquid, hereby to raise the velocity of dispersal. In order to subscribe to the above mechanism, the same gravitational exchange function, which represents the pull, should not only play the role of building up the kinetic energy as an acceleration, but also of depleting the potential energy as a stabilizing force, with equal but opposite amount in the system of equations for the balance between the kinetic energy and the potential energy. Since the spectrum of the kinetic energy is in the production subrange as a result of the gravitational pull, the friction will be neglected by writing $\gamma = 0$ in Eq. (131). In addition, we omit the molecular dissipation functions in both equations, reducing the system of equations (131) and (133) to

$$-\Gamma^{\circ} + \eta' J^{\circ} = -\Gamma + \varepsilon \tag{163}$$

$$\Gamma^{\circ} + \eta' J^{\circ}_{w} = \Gamma + \varepsilon_{w} \tag{164}$$

In order to discern the energy flows across each of the two spectra more conveniently, we rewrite (163) and (164) in their differential form

$$-\frac{(gk)^2 H}{\omega'} + \eta' \frac{dJ^\circ}{dk} + \frac{d\eta'}{dk} J^\circ = 0 \qquad (165)$$

$$\frac{(gk)^2 H}{\omega'} + \eta' \frac{dJ_w^\circ}{dk} + \frac{d\eta'}{dk} J_w^\circ = 0 \ , \qquad (166)$$

describing the energy flows across each of the two spectra which are in two different regimes of development. The spectrum of kinetic energy is in its early stage of development in k-space, in view of its regime of production. Therefore its vorticity is not yet formed, permitting the approximation $J^\circ \cong o$. On the other hand, the spectrum of potential energy is in its later stage of development in k-space, in view of its stabilization by gravity. Therefore the large wave numbers controlling the potential spectrum bring its vorticity to saturation, permitting the approximation $J_w^\circ \cong J_w$, and at the same time rendering η' negligible. As a result, the system of equations (165) and (166) simplify to

$$-\frac{(gk)^2 H}{\omega'} + \eta' \frac{dJ^\circ}{dk} = 0 \qquad (167)$$

$$\frac{(gk)^2 H}{\omega'} + \frac{d\eta'}{dk} J_w = 0 \ . \qquad (168)$$

The system of integro-differential equations (167) and (168) yield the solutions

$$F = J_w k^{-3} \qquad (169)$$

$$H = B_g k^{-5} \ , \qquad (170)$$

with

$$B_g = \left(J_w / g^* \right)^2 \tag{171}$$

XIII. SIMILARITY THEORY

Unstable surfaces refer to turbulent surfaces of a heavy liquid above a light one, and turbulence is generated as a result of Taylor instability. Stable surfaces refer to the surface of a heavy liquid below a light fluid, e.g. a sea surface.

We shall summarize, in Table 1, the results of the spectral distributions obtained from the present analytical theory. This gives us an opportunity of outlining their fundamental mechanisms and governing parameters, and of supplying an additional dimensional theory of the spectral laws. For the sake of abbreviation, we shall omit the surface tension in the dimensional considerations. The numerical coefficients will also not be written.

We distinguish the following subranges:

A. Production by gravitational acceleration

This subrange exists for unstable surfaces and is absent for stable surfaces.

The frictionless case is governed by the parameter g, and on dimensional considerations can find the spectra

$$F = g k^{-2}, \qquad H = k^{-3} \tag{172}$$

for the kinetic energy and the surface elevation, respectively.

The frictional case has a second parameter, which is the frictional wave number

$$k_\gamma = \gamma^2 / g . \tag{173}$$

Since the elevation of the surface is opposed by the friction γ to its first power, we find

$$F = g k^{-2} , \qquad H = (k_\gamma / k)^{\frac{1}{2}} k^{-3} . \tag{174}$$

B. Inertia

The spectral laws in the inertia subrange are independent of the gravitational effects, and are, therefore, common to stable and unstable surfaces.

The frictionless laws are governed by the parameters ε and ε_ζ, and agree with the Kolmogoroff laws:

$$F = \varepsilon^{2/3} k^{-5/3} , \qquad H = \varepsilon_\zeta \varepsilon^{-1/3} k^{-5/3} . \tag{175}$$

Their dimensional derivations are well known and will not be repeated here.

If the friction is dominant, the kinetic energy is transferred across the spectrum to secure a balance between the friction and the saturated vorticity. The governing parameter is

$$\Omega_\gamma = J / \gamma , \tag{176}$$

and a dimensional consideration with this parameter gives a spectrum

$$F = \Omega_\gamma^2 k^{-3} \tag{177}$$

Since the surface spectrum should be proportional to ε_ζ according to (135), a dimensional analysis using the parameters Ω_γ and ε_ζ yields

$$H = (\varepsilon_\zeta / \Omega_\gamma) k^{-1} \tag{178}$$

C. Frictionless eddy dissipation by gravitational pull

This mechanism of wave dissipation controls a stable surface only. The governing parameters are the potential vorticity J_w and the gravity g. Since the build-up of the kinetic energy occurs at the expense of the potential energy with a vorticity J_w, the spectrum of the kinetic energy is

$$F = J_w k^{-3} \tag{179}$$

By balancing the gravitational acceleration with the nonlinear eddy transfer in Eq. (167), we find

$$\zeta = u^2 / g \tag{180}$$

Upon substituting (179), we find, from (180), the surface spectrum

$$H = (J_w / g)^2 k^{-5}, \tag{181}$$

on a dimensional argument.

D. Molecular dissipation

The frictionless laws

$$F = \mu_\nu^2 k^{-7} \tag{182}$$

$$H = \mu_\lambda \mu_\nu k^{-7} \tag{183}$$

agree with the Heisenberg theory for viscous dissipation.

In order to transform those viscous laws into inviscid, but frictional, laws, a wave number

$$k^* = (\gamma/\nu)^{\frac{1}{2}} = (\mu_\nu / \Omega_\gamma)^{\frac{1}{2}} \tag{184}$$

has to be introduced, to convert the formulas (183) and (184) into

$$F = \mu_\nu^2 k^{-7} (k/k^*)^m \tag{185}$$

$$H = \mu_\lambda \mu_\nu k^{-7} (k/k^*)^n \tag{186}$$

We see that we must have m = 4 and n = 2 for the formulas (185) and (184) to be inviscid, entailing

$$F = \Omega_\gamma^2 k^{-3} , \quad H = \mu_\lambda \Omega_\gamma k^{-5} \tag{187}$$

We conclude that the above dimensional considerations enable the reproduction of the results of the analytical theory.

XIV. COMPARISON WITH OBSERVATIONS

The present cascade theory has predicted a wavenumber spectrum k^{-5} for the fluctuations of the surface in the gravitational subrange (170). This law differs from the dimensional law

$$H(k) = \text{const}\ k^{-3}, \tag{188}$$

as proposed by Phillips[4] for sea-surface.

By introducing a streaming velocity u_s of the surface as a reference velocity, we can make a change of variables from k into a new variable

$$n = k u_s \tag{189}$$

having the dimension of a frequency, and convert (170) into a spectral distribution

$$H(n) = \beta g^2 n^{-5}, \tag{190}$$

normalized to

$$\frac{1}{2}\overline{\zeta^2} = \int_0^\infty dn\ H(n) \tag{191}$$

with

$$\beta = c_1 \left(J_w u_s^2 / g^2 \right)^2 \tag{192}$$

The analytical law of spectrum (170) is now brought close to the dimensional formula. However, this similarity is only apparent, because (170) expresses a wavenumber spectrum, in spite of the presence of n having the dimension of a frequency, while (4) was proposed as a frequency law. Moreover, the dimensionless coefficient β, which is determined in the analytic law, could not have been determined in the dimensional law (4).

Several observations have been reported[4,6,7,10], measuring the spatial elevation fluctuations as a function of time as they are conveyed to the point of the probe by a streaming velocity u_s. If u_s is large in comparison with the phase velocity, $c(k\ k_m)$, of the elevation fluctuations in the gravitational subrange, their spatial scales past the point of measurements will be observed as fluctuations in time, with a frequency (189), according to the Taylor hypothesis. Here k_m is the wavenumber corresponding to the peak of the spectrum preceding the gravitational subrange. Although the average phase velocity can be sufficiently large, invalidating the Taylor hypothesis, the small scales of the gravitational subrange may give a small enough $c(k\ k_m)$ as to satisfy the Taylor hypothesis, thus enabling the comparison between measurements and the theoretical prediction based upon (190) and (192).

It will be convenient to express J_w in (192) in terms of the mean square slope J_ζ, which is a measurable quantity. For this purpose and with the use of (120), we rewrite (118) in the form

$$H + \frac{J_w}{2g^2}\frac{d\eta'^2}{dk} = 0 \qquad (193)$$

which, through (130), is transformed into

$$H = \frac{J_w}{g^2} k^{-3} F, \tag{194}$$

and is integrated to give

$$J_\zeta = 2\int_0^\infty dk\, k^2 F = \frac{2J_w}{g^2}\int_0^\infty dk\, F \tag{195}$$

The kinetic energy spectrum derives its energy mostly from its production subrange, with a spectrum (169). Therefore, without much error, we can substitute for F from (169) with a cutoff wavenumber k_0 at the outerscale $2\pi/k_0$, reducing (195) to

$$J_\zeta = (J_w/gk_0)^2, \tag{196}$$

or equivalently

$$J_w = J_\zeta^{\frac{1}{2}} g k_0 \tag{197}$$

A further substitution into (192) yields

$$\beta = \alpha J_\zeta \tag{198}$$

with

$$\alpha = (k_0 u_*^2/g)^2 \cong 0.11 \tag{199}$$

This value is computed by taking the empirical estimates[8-10]:

$$k_0 = g/u'^2,$$

$$u_s = 0.87\, u_*, \quad u' = 1.5\, u_*, \tag{200}$$

where u_* is the frictional velocity, and u' is the standard deviation of velocity fluctuations.

It is to be noted that the mean square slope J_ζ has been measured experimentally and reported by Wu[11]. In Fig. 1 we have plotted the experimental values of β based upon the spectral data of Volkov[6], and found them in good agreement with the theoretical predictions (198) and (199).

XV. CONCLUSIONS

The spectral laws in (172) are valid for the production subrange in an unstable surface, and are not valid in a stable surface as claimed by Phillips, see (3) and (4).

A stable surface has the κ^{-5} law (170) in the eddy dissipation or gravitational subrange. The same power (143) holds in the molecular dissipation subrange. If the coefficients in the two formulas do not differ much, the two laws of the same power will appear in a continuous succession. This explains why the 5th power law is so easily found on sea-surface turbulence.

Under the circumstances where the Taylor hypothesis is valid, the analytical law can be brought to a form (190) which has an appearance analogous to the dimensional law (4) proposed by Phillips[4]. The coefficient β, which was proposed as a universal constant in the dimensional theory, becomes a function of the dimensionless sea-surface slope, as predicted by

(198). The agreement between the measured relation for β and the theoretical prediction (198) is shown satisfactory in Fig. 1.

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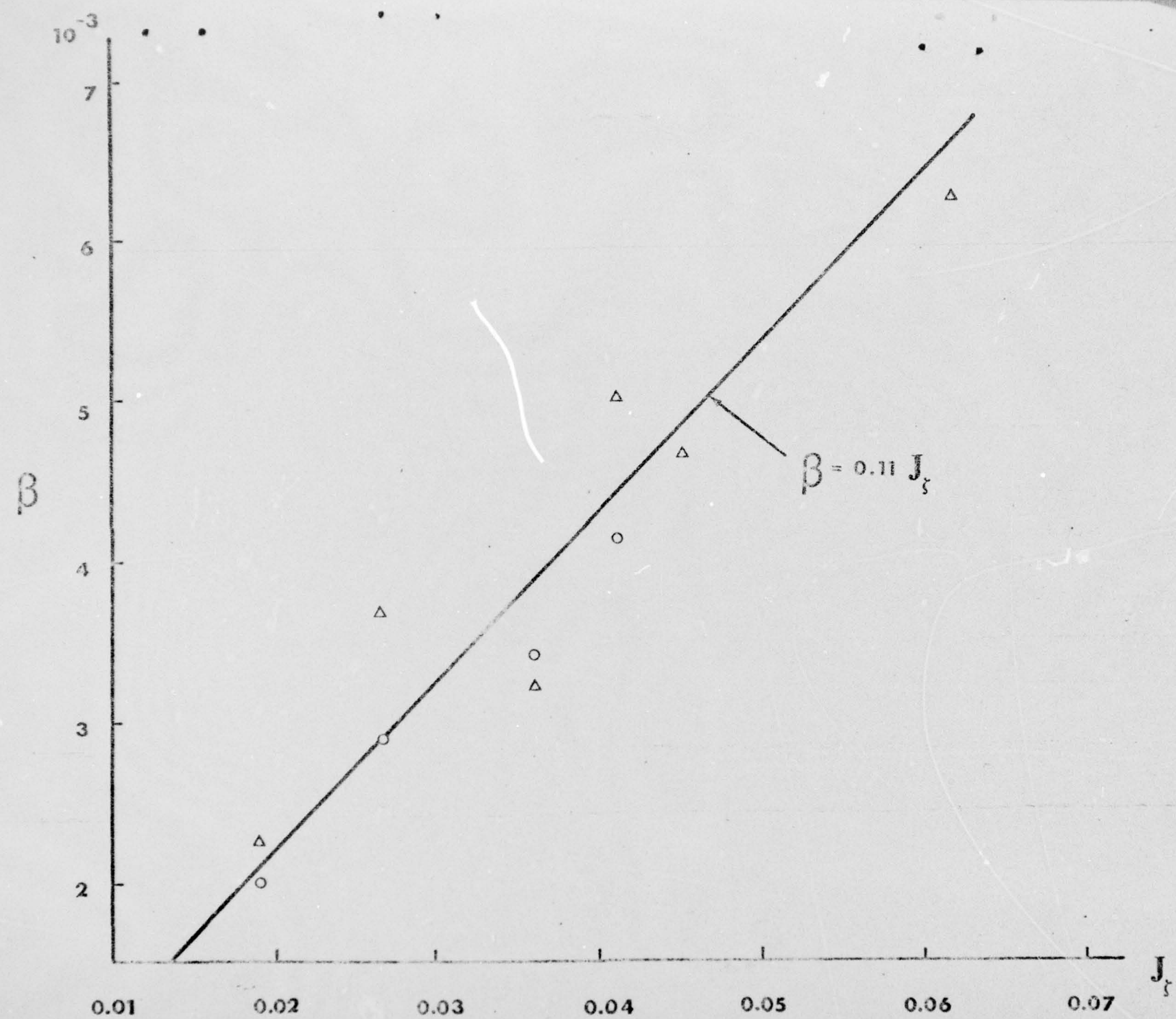
Figure Legend

Fig. 1 Variation of coefficient β in Eq. (192) with mean square slope J_ζ. The experimental points are obtained by using the spectral data reported by Volkov[6]. The solid line represents the theoretical prediction of Eqs. (198) and (199).

Subranges / Surfaces		Production by gravitational acceleration: frictionless	Production by gravitational acceleration: frictional	Inertia: frictionless	Inertia: frictional	Eddy dissipation by gravitational pull (frictionless and inviscid)	Molecular dissipation: frictionless	Molecular dissipation: frictional
(unstable surface) turbulence from Taylor instability		$F = gk^{-2}$ $H = k^{-3}$	$F = gk^{-2}$ $H = (k/k_\gamma)^{-\frac{1}{2}}k^{-3}$	$F = \epsilon^{2/3}k^{-5/3}$ $H = \epsilon_\zeta \epsilon^{-1/3}k^{-5/3}$	$F = \Omega_\gamma^2 k^{-3}$ $H = (\epsilon_\zeta/\Omega_\gamma)k^{-1}$	absent	$F = \mu_\nu^2 k^{-7}$ $H = \mu_\lambda \mu_\nu k^{-7}$	$F = \Omega_\gamma^2 k^{-3}$ $H = \mu_\lambda \Omega_\gamma k^{-}$[illegible]
(stable surface) sea surface turbulence		absent				$F = J_w k^{-3}$ $H = k_g^2 k^{-5}$		
controlling parameters	F	g	g	ϵ	$\Omega_\gamma = J/\gamma$	J_w	$\mu_\nu = J/\nu$	Ω_γ
	H		$k_\gamma = \gamma^2/g$	ϵ, ϵ_ζ	$\epsilon_\zeta, \Omega_\gamma$	$k_g = J_w/g$	$\mu_\nu, \mu_\lambda = J_\zeta/\lambda$	$\mu_\lambda \cdot \Omega_\gamma$

Table 1. Spectral distributions F and H for stable and unstable surfaces


10^{-3}
7
6
5
4
3
2
β
$\beta = 0.11\ J_\zeta$
0.01
0.02
0.03
0.04
0.05
0.06
0.07
J_ζ